2D XY Model RG Flows for the Intrepid Student

Abdulai Gassama
Brown University
PHYS2470

December 19, 2024

Abstract

In this paper, I thoroughly derive the renormalization group (RG) flows for the 2D XY model starting from first principles. I begin with the concept of vortices in the 2D XY model and proceed to establish dualities with Coulomb gases and the Sine-Gordon equation. By considering topological excitations (vortices) and their interactions, and mapping these to well-known equivalent models, I make explicit the connection to Coulomb gas partition functions and Sine-Gordon field theory. Finally, I extract the RG flow equations that describe the behavior of the coupling constants as the system is viewed at larger length scales. In the appendices, I provide every intermediate step and derivation in full detail, with no steps omitted.

Road to the Main Results

1. XY Model and Vortices: The 2D XY model Hamiltonian is

$$H_{\mathrm{XY}} = -J \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j).$$

At low temperatures (T), it can be approximated by a free Gaussian theory in terms of a continuous field $\theta(\mathbf{x})$. Vortices appear as topological defects with phase winding. A single vortex configuration costs energy proportional to $\ln(L/a)$.

2. Coulomb Gas Representation: Vortices in the XY model can be represented as charges in a 2D Coulomb gas. This maps the XY partition function onto that of a neutral plasma of point charges with logarithmic interactions.

3. Sine-Gordon Relation: Introducing a cosine perturbation in a scalar field theory (the Sine-Gordon model) creates particle-antiparticle pairs analogous to vortices. By identifying parameters, one shows that the XY model's vortex sector is equivalent to a Sine-Gordon theory at a certain coupling.

4. RG Flow Equations: Integrating out short-distance degrees of freedom (small vortex-antivortex pairs) leads to the Kosterlitz-Thouless RG equations:

$$\frac{dK^{-1}}{d\ell} = 4\pi^3 y^2, \quad \frac{dy}{d\ell} = (2 - \pi K)y,$$

where $K = \beta J$ and y is the vortex fugacity. The KT transition occurs at $K_c = 2/\pi$.

Contents

1 Establishing the XY Model

The 2D XY model is characterized by planar spins $\vec{S}_i = (\cos(\theta_i), \sin(\theta_i))$ on a lattice as seen in Figure 1. Its Hamiltonian is:

$$H_{\mathrm{XY}} = -J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j = -J \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j).$$

The partition function is:

$$\mathcal{Z} = \int \prod_{i=1}^{N} \frac{d\theta_i}{2\pi} e^{-\beta H_{\mathrm{XY}}}.$$

At low temperature, we approximate for small angle differences:

$$-\beta H_{XY} \approx -\frac{K}{2}\int d^2x(\nabla\theta)^2 + \ldots \quad \text{where } K = \beta J.$$

Vortices are topological configurations in the XY model. Consider a vortex configuration defined by:

$$\phi(r,\theta) = q\theta.$$

A vortex has the property:

$$\oint \vec{\nabla}\phi(r,\theta)\cdot d\vec{l} = 2\pi q.$$

In polar coordinates:

$$\vec{\nabla}\phi = \hat{\theta}\frac{q}{r}.$$

The energy of a single vortex configuration of core size a and system size L is:

$$H = \frac{J}{2}\int_a^L 2\pi r dr\left(\frac{q^2}{r^2}\right) = q^2 J\pi \ln\frac{L}{a}.$$

For a single vortex of charge $q = 1$, the cost grows logarithmically with the system size.

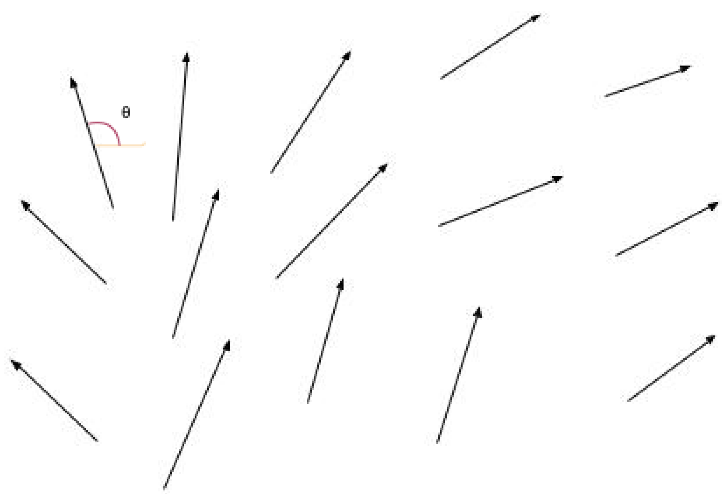

Figure 1: A vortex configuration in the 2D XY model. The spin orientation angle θ winds by 2π around the center.

In Cartesian coordinates (x, y), a single vortex configuration can be written as:

$$\phi(r) = q\arctan\frac{y}{x}, \quad \vec{\nabla}\phi(r) = q\hat{z}\times\frac{\vec{r}}{r^2}.$$

For a vortex-antivortex pair:

$$\vec{\nabla}\phi(\vec{r}) = \hat{z}\times\left(\frac{\vec{r}-\vec{r}_+}{|\vec{r}-\vec{r}_+|^2} - \frac{\vec{r}-\vec{r}_-}{|\vec{r}-\vec{r}_-|^2}\right).$$

At large distances r, the field due to a pair decays as $\sim \frac{1}{r^2}$.

Since $\nabla^2 \log(r) = 2\pi\delta(r)$, vortex configurations can be thought of as 2D Coulomb charges, with the vortex acting like a charge sourcing a logarithmic potential.

2 Establishing Relation with the Coulomb Gas

Consider the Gaussian integral representation:

$$Z[J] = \int D_\phi \exp\left(-\frac{1}{2}\int dx\, \phi[-K\nabla^2]\phi + \int dx\, J(x)\phi(x)\right).$$

After completing the square and integrating out ϕ, we get:

$$Z[J] = Z[0]\exp\left\{\left(\frac{1}{2}\int dxdy\, J(x)\hat{O}^{-1}(x-y)J(y)\right)\right\},$$

where $\hat{O} = -K\nabla^2$.

We can identify the two-point function:

$$\langle\phi(x)\phi(y)\rangle = \hat{O}^{-1}(x-y).$$

For the XY model at low temperature:

$$\langle\theta(r)\theta(0)\rangle = -\frac{1}{2\pi K}\ln\frac{r}{L}.$$

A configuration of vortices $\phi = \sum_i m_i \log|\vec{x}-\vec{x}_i|$ leads to:

$$H = \frac{J}{2}\int d^2x(\nabla\phi)^2 = \sum_{i,j} m_i m_j \pi J \ln\frac{|\vec{x}_i-\vec{x}_j|}{a}.$$

This maps onto a 2D Coulomb gas of charges $m_i = \pm 1$ with a logarithmic interaction:

$$\mathcal{Z}_{XY} \approx \left(\int D_\phi e^{-\frac{K}{2}\int d^2x(\nabla\phi)^2}\right)\left(\sum_N \frac{y^N}{(N/2)!^2}\int\prod_i d^2x_i \exp\left(4\pi^2 K\sum_{i<j} m_i m_j C(\vec{x}_i-\vec{x}_j)\right)\right),$$

where

$$C(\vec{x}_i-\vec{x}_j) = \frac{\log|\vec{x}_i-\vec{x}_j|}{2\pi}, \quad y = e^{-\beta\epsilon^0_\pm}.$$

Hence, the vortex sector of the XY model can be represented as a 2D Coulomb gas of ± 1 charges with fugacity y.

3 Establishing the Sine-Gordon Relation

The Sine-Gordon model:

$$\beta H_{\text{sg}} = \int d^2x \left[\frac{K'}{2} (\nabla\varphi)^2 - g\cos(\varphi) \right].$$

Expanding $e^{g\cos\varphi}$ in a series:

$$e^{g\cos\varphi} = \sum_{N=0}^{\infty} \frac{(g/2)^N}{N!} \int \prod_{i=1}^{N} d^2x_i \sum_{\{m_i=\pm1\}} \exp\left(i \sum_i m_i \varphi(\vec{x}_i) \right).$$

Gaussian averaging over φ gives a similar Coulomb gas representation:

$$\langle \varphi(\vec{x})\varphi(\vec{y}) \rangle = -\frac{1}{2\pi K} \ln|\vec{x} - \vec{y}|.$$

By matching parameters, one can show that the Sine-Gordon model at small g maps onto the same Coulomb gas representation as the XY vortices:

$$\frac{\mathcal{Z}_{\text{sg}}}{\mathcal{Z}^0_{\text{sg}}} = \mathcal{Z}_Q, \text{ for which } \mathcal{Z}^0_{sg} \text{ is without perturbation}$$

and now we can identify:

$$K_{\text{XY}} = \frac{1}{8\pi^2 K_{\text{sg}}}.$$

Thus, the XY model, Coulomb gas, and Sine-Gordon model are all dual representations of each other.

4 Deriving the RG Flows

The scaling dimension of a vortex operator $e^{im\phi}$ in the Gaussian model is $\pi K m^2$. For a single vortex ($m = 1$), if $\pi K < 2$, vortices are relevant operators under RG; if $\pi K > 2$, they are irrelevant.

Integrating out short-distance vortex-antivortex pairs generates flow equations for K and y. The standard Kosterlitz-Thouless RG equations are:

$$\frac{dK^{-1}}{d\rho} = 4\pi^3 y^2, \quad \frac{dy}{d\rho} = (2 - \pi K)y,$$

where ρ is the RG scale (log of length scale).

These flow equations show a line of fixed points at $y = 0$ and $K > 2/\pi$ (low-temperature, quasi-long-range order) and a critical point at $K = 2/\pi$ where the system undergoes the Kosterlitz-Thouless transition. For $K < 2/\pi$, vortices proliferate, destroying quasi-long-range order.

An invariant combination under RG is $x^2 - \pi^4 y^2$, where $x = K^{-1} - \pi/2$. This indicates hyperbolic flow trajectories in the (K, y) plane as seen in Figure 2.

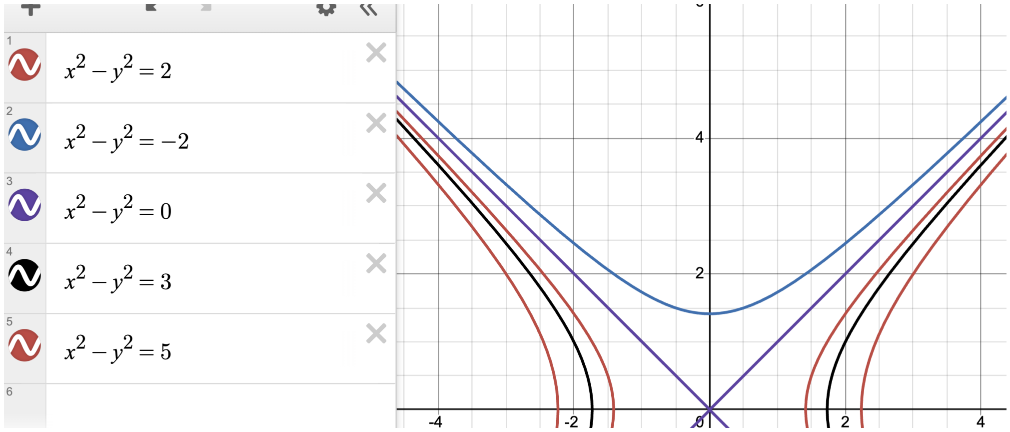

Figure 2: Representative RG flows for the 2D XY model, depicting the stable fixed points and the Kosterlitz-Thouless transition line. The flow lines show how the coupling constants evolve as one changes the length scale. They also describe a phase transition driven by the unbinding of vortex-antivortex pairs. For $K > 2/\pi$, the system exhibits quasi-long-range order (spin correlations decay as a power law), while for $K < 2/\pi$, vortex proliferation leads to exponential decay of correlations and disordered phase.

A Appendix: Complete Derivations

In this appendix, I present every step of the derivations mentioned in the main text, without skipping any intermediate reasoning.

A.1 Important Math Proof

Mini math prove involving a Gaussian

$$
\begin{aligned}
\langle \exp\{ix\} \rangle &= \int dx \exp\{ix\} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} \frac{1}{\sqrt{2\pi\sigma^2}} \\
&= \int \frac{dx}{\sqrt{2\pi\sigma^2}} \exp\left\{\left[ix - \frac{x^2}{2\sigma^2}\right]\right\} \\
&= \int \frac{dx}{\sqrt{2\pi\sigma^2}} \exp\left\{\left[\frac{1}{2\sigma^2}\left[2\sigma^2 ix - x^2\right]\right]\right\} \\
&= \int \frac{dx}{\sqrt{2\pi\sigma^2}} \exp\left\{\left[\frac{-1}{2\sigma^2}\left[-2\sigma^2 ix + x^2\right]\right]\right\} \\
&= \int \frac{dx}{\sqrt{2\pi\sigma^2}} \exp\left\{\left[\frac{-1}{2\sigma^2}\left[-2\sigma^2 ix + x^2 + \sigma^4 - \sigma^4\right]\right]\right\} \\
&= \int \frac{dx}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}\left[[x - \sigma i]^2 + \sigma^4\right]\right\} \\
&= \int \frac{dx}{\sqrt{2\pi\sigma^2}} \exp\left\{\left[\frac{-1}{2\sigma^2}[x - \sigma i]^2 - \frac{\sigma^2}{2}\right]\right\} \\
&= \underbrace{\int \frac{dx}{\sqrt{2\pi\sigma^2}} \exp\left\{\left[-\frac{x^2}{2\sigma^2}\right]\right\}}_{1} \exp\left\{\left[-\frac{\sigma^2}{2}\right]\right\} \\
&= \exp\left\{-\frac{1}{2}\sigma^2\right\}.
\end{aligned}
$$

Keep in mind we are allowed to shift a contour integrations if the function holomorphic, and it is for a gaussian since

$$
f(z) = \exp\left\{\left(iz - \frac{z^2}{2\sigma^2}\right)\right\}
$$

is analytic everywhere and no poles exists; and $f(z)$ goes to 0 on real axis as $|z| \to \infty$. Thus $x \to x + i\sigma$ is valid. For a gaussian distribution with a mean of zero the variance is

$$
\begin{aligned}
Var(X) &= \left\langle X^2 \right\rangle - \underbrace{\langle X \rangle^2}_{\text{mean squared}} \\
Var(X) &= \left\langle X^2 \right\rangle,
\end{aligned}
$$

so we can say $\langle x^2 \rangle = \sigma^2$ and

$$
\langle \exp\{ix\} \rangle = \exp\left\{-\frac{1}{2}\left\langle x^2 \right\rangle\right\}.
$$

A.2 A1. XY Model Setup and Vortex Definition

The 2D XY model is defined on a 2D lattice with spins:

$$
\vec{S}_i = (\cos(\theta_i), \sin(\theta_i)).
$$

The Hamiltonian is:

$$H_{\text{XY}} = -J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j = -J \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j).$$

At low temperatures, where θ_i varies slowly between neighboring sites, we can go to a continuum limit. Let the lattice spacing be a, and define a continuous field $\theta(\mathbf{x})$. For small $(\theta_i - \theta_j)$:

$$\cos(\theta_i - \theta_j) \approx 1 - \frac{(\theta_i - \theta_j)^2}{2}.$$

For a 2D continuum:

$$-\beta H_{\text{XY}} \approx -\frac{K}{2} \int d^2x\, (\nabla\theta)^2, \quad \text{where } K = \beta J.$$

For high temperature note that,

$$e^{-\beta H_{\text{XY}}} = \left[1 + K\cos\left(\theta_i - \theta_j\right) + \mathcal{O}(K^2)\right].$$

A vortex configuration is defined by:

$$\oint \nabla\theta \cdot d\ell = 2\pi q,$$

where q is an integer (vortex charge). For a vortex at the origin with $q = 1$:

$$\theta(r, \phi) = \phi,$$

in polar coordinates (r, ϕ).

The gradient of this configuration is:

$$\nabla\theta = \frac{\hat{\phi}}{r}.$$

In Cartesian coordinates (x, y), a single vortex can be written as:

$$\theta(x, y) = \arctan\left(\frac{y}{x}\right).$$

Differentiating:

$$\frac{\partial\theta}{\partial x} = \frac{-y}{x^2 + y^2}, \quad \frac{\partial\theta}{\partial y} = \frac{x}{x^2 + y^2}.$$

Thus:

$$\nabla\theta = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right) = \hat{z} \times \frac{\mathbf{r}}{r^2}.$$

The energy of a single vortex configuration (core size a, system size L) in the Gaussian approximation:

$$H = \frac{J}{2} \int d^2x (\nabla\theta)^2.$$

Since $(\nabla\theta)^2 = \frac{1}{r^2}$ for a vortex

$$H = \frac{J}{2} \int_a^L 2\pi r\, dr \frac{1}{r^2} = J\pi \ln\frac{L}{a}.$$

A.3 A2. Vortex-Antivortex Pair and Field Configuration

For a vortex at $\mathbf{r}_+$ and an antivortex at $\mathbf{r}_-$, the combined field is:

$$\theta(\mathbf{r}) = \arctan \frac{y - y_+}{x - x_+} - \arctan \frac{y - y_-}{x - x_-}.$$

Now to look into

$$\vec{\nabla}\phi = \vec{\nabla}\phi_+ + \vec{\nabla}\phi_-.$$

We've seen

$$\begin{aligned}\vec{\nabla}\phi_+ &= q\hat{z} \times \frac{\vec{r}}{r^2} \\ &= q\hat{z} \times \vec{\nabla} \ln |\vec{r}| \\ &= q\hat{z} \times \hat{r}\frac{1}{r} \\ &= q\hat{z} \times \frac{\vec{r}}{r^2}\end{aligned}$$

So the gradient:

$$\nabla\theta(\mathbf{r}) = \hat{z} \times \left(\frac{\mathbf{r} - \mathbf{r}_+}{|\mathbf{r} - \mathbf{r}_+|^2} - \frac{\mathbf{r} - \mathbf{r}_-}{|\mathbf{r} - \mathbf{r}_-|^2} \right).$$

$$\vec{\nabla}\phi = -\left[\vec{\nabla} \times \hat{z} \ln \frac{|\vec{r} - \vec{r}_+|}{|\vec{r} + \vec{r}_+|} \right]$$

Say that $r_+ = -\frac{d}{2}$ and $r_- = \frac{d}{2}$, also keep in mind binomial expansion $(1+x)^{-1} = 1 - x + x^2 + \mathcal{O}(x^3)$ see that

$$\begin{aligned}\ln \frac{|\vec{r} - \vec{d}/2|}{|\vec{r} + \vec{d}/2|} &= \ln \frac{\sqrt{(\vec{r}^2 - \vec{r} \cdot \vec{d} + (\vec{d})^2}}{\sqrt{(\vec{r})^2 + \vec{r} \cdot \vec{d} + \vec{d}/4}} \\ &= \frac{1}{2} \ln \frac{(\vec{r})^2 - \vec{r} \cdot \vec{d} + (\vec{d})^2/4}{(\vec{r})^2 + \vec{r} \cdot \vec{d} + \vec{d}/4} \\ &= \frac{1}{2} \ln \frac{(\vec{r})^2 - \vec{r} \cdot \vec{d}}{(\vec{r})^2 + \vec{r} \cdot \vec{d}} \\ &= \frac{1}{2} \ln \left(\frac{r^2}{r^2} \left[\frac{1 - \vec{r} \cdot \vec{d}/r^2}{1 + \vec{r} \cdot \vec{d}/\vec{r}^2} \right] \right) \\ &= \frac{1}{2} \ln \left[\left(1 - \vec{r} \cdot \vec{d}/r^2\right) \left(1 - \vec{r} \cdot \vec{d}/r^2\right) \right] \\ &= \frac{1}{2} \ln \left[1 - \frac{2\vec{r} \cdot \vec{d}}{r^2} + \left(\frac{\vec{r} \cdot \vec{d}}{r^2} \right)^2 \right] \\ &\approx \frac{1}{2} \ln \left[1 - \frac{2\vec{r} \cdot \vec{d}}{r^2} \right].\end{aligned}$$

For a vortex-antivortex pair:

$$\begin{aligned}
\vec{\nabla}\phi = \vec{\nabla}\phi_+ + \vec{\nabla}\phi_- &= \hat{z}\times\left(\frac{\vec{r}-\vec{r}_+}{|\vec{r}-\vec{r}_+|^2} - \frac{\vec{r}-\vec{r}_-}{|\vec{r}-\vec{r}_-|^2}\right) \\
&\approx \hat{z}\times q\left(\frac{(\vec{r}-\vec{r}_+)}{r^2}\left(1+\frac{2\vec{r}\cdot\vec{r}_+}{r^2}+\mathcal{O}\left(\frac{1}{r^2}\right)\right) - \frac{(\vec{r}-\vec{r}_-)}{r^2}\left(1+\frac{2\vec{r}\cdot\vec{r}_-}{r^2}+\mathcal{O}\left(\frac{1}{r^2}\right)\right)\right) \\
&= \hat{z}\times\left(\frac{\vec{r}_+ + \vec{r}_-}{r^2} + \frac{2\vec{r}\left[\vec{r}\cdot(\vec{r}_+ + \vec{r}_-)\right])}{r^4} + \mathcal{O}\left(\frac{1}{r^2}\right)\right) \\
&\sim \frac{1}{r^2}.
\end{aligned}$$

At large distances r, this decays as $1/r^2$, ensuring no net circulation far away.

A.4 A3. Coulomb Gas Representation

$$Z[J] = \int D_\phi \exp\left\{-\frac{1}{2}\int dx\phi\left[-K\nabla^2\phi\right] + \int dx J(x)\phi(x)\right\}.$$

Say $\hat{O} \equiv -K\nabla^2$, $\phi J = \frac{1}{2}\phi\hat{O}\hat{O}^{-1}J + \frac{1}{2}\phi\hat{O}\hat{O}^{-1}J$, and $[\phi, J] = 0$ so $\frac{1}{2}\phi\hat{O}\hat{O}^{-1}J = \frac{1}{2}J\hat{O}\hat{O}^{-1}\phi$.

Then

$$\begin{aligned}
Z[J] &= \int D_\phi \exp\left\{-\frac{1}{2}\int dx\phi\hat{O}\phi + \int dxJ\phi\right\} \\
&= \int D_\phi \exp\left\{-\frac{1}{2}\int dx\phi\hat{O}\phi + \int dxJ\phi + \frac{1}{2}J\hat{O}^{-1}J - \frac{1}{2}J\hat{O}^{-1}J]\right\} \\
&= \int D_\phi \exp\left\{-\frac{1}{2}\int dx\left(\phi-\hat{O}^{-1}J\right)\hat{O}\left(\phi-\hat{O}^{-1}J\right) + \frac{1}{2}\int dxJ\hat{O}^{-1}J\right\} \\
&= \int \underbrace{D_\eta}_{\text{No Jacobian factor}} \exp\left\{-\frac{1}{2}\int dx\eta\hat{O}\eta + \frac{1}{2}\int dxJ\hat{O}^{-1}J\right\}.
\end{aligned}$$

For which $\phi \to \phi - \hat{O}^{-1}J$ as the physics remains the same. So now the correlation function can be rewritten as $Z[J] = Z[0]\exp\left\{\left[\frac{1}{2}\int dxdyJ(x)\hat{O}^{-1}J(y)\right]\right\}$.

$$
\begin{aligned}
\langle\phi(x)\phi(y)\rangle &= \left.\frac{\partial^2 \ln Z[J]}{\partial J(x)\partial J(y)}\right|_{J=0} \\
&= \frac{\partial}{\partial J(y)}\left.\left[\left[\frac{1}{2}\int dxdy\delta(x-x')\hat{O}(x-y)J(y) + \frac{1}{2}\int dxdyJ(x)\hat{O}(x-y)\delta(y-x')\right]\right]\right|_{J=0} \\
&= \frac{\partial}{\partial J(y)}\left.\left[\left[\frac{1}{2}\int dxdy\delta(x-x')\hat{O}(x-y)J(y) + \frac{1}{2}\int dxdyJ(x)\hat{O}(x-y)\delta(y-x')\right]\right]\right|_{J=0} \\
&= \left.\left[\frac{1}{2}\int dxdy\delta(x-x')\hat{O}^{-1}(x-y)\delta(y-y') + \frac{1}{2}\int dxdy\delta(x-y')\hat{O}^{-1}(x-y)\delta(y-x')\right]\right|_{J=0} \\
&= \left[\frac{1}{2}\hat{O}^{-1}(x'-y') + \frac{1}{2}\hat{O}^{-1}(y'-x')\right] \\
&= \hat{O}^{-1}(x'-y').
\end{aligned}
$$

So now we see the relation

$$
\begin{aligned}
\hat{O}\langle\phi(x)\phi(y)\rangle &= \delta(x-y) \\
\langle\theta(r)\theta(0)\rangle &= \frac{-1}{2\pi K}\ln r/L = G(r)
\end{aligned}
$$

where L is the UV cutoff and $G(r)$ is a green's function. Also in general we know $\nabla^2\log(r) = 2\pi\delta(r)$, so through superposition

$$
\phi = \sum_i m_i \log|\vec{x} - \vec{x}_i|
$$

which is the potential due to a set of vortices. m_i = topological charge. It also follow that $\nabla^2\phi = 2\pi\sum_i m_i\delta^2(\vec{x}-\vec{x}_i)$ For analogy purposes we will be looking at the low temperature regime,

$$
\begin{aligned}
\beta H_{\text{XY,low}} &= \frac{K}{2}\int d^2x(\nabla\theta)^2 \\
&= \frac{K}{2}\int d^2x\left[(\vec{\nabla}\varphi)^2 - 2\vec{\nabla}\varphi\cdot\left(\vec{\nabla}\times\hat{z}\phi\right) + \left(\vec{\nabla}\times\hat{z}\varphi\right)^2\right]
\end{aligned}
$$

The coupled term equals zero

$$
\begin{aligned}
-2\int d^2x\vec{\nabla}\varphi\cdot\left(\vec{\nabla}\times\hat{z}\phi\right) &= 2\int d^2x\left[\frac{\partial\phi}{\partial y}\frac{\partial\varphi}{\partial x}-\frac{\partial\phi}{\partial x}\frac{\partial\varphi}{\partial y}\right]\\
&= 2\left[\frac{\partial\phi}{\partial y}\varphi\bigg| - \int d^2x\frac{\partial\varphi}{\partial x}\frac{\partial^2\phi}{\partial y^2} - \int d^2x\frac{\partial\phi}{\partial x}\frac{\partial\varphi}{\partial y}\right]\\
&= 2\left[\frac{\partial\phi}{\partial y}\varphi\bigg| - \int d^2x\frac{\partial\varphi}{\partial x}\frac{\partial^2\phi}{\partial y^2} - \frac{\partial\phi}{\partial x}\varphi\bigg| - \int d^2x\frac{\partial\varphi}{\partial y}\frac{\partial^2\phi}{\partial x^2}\right]\\
&= 2\Bigg[\underbrace{\frac{\partial\phi}{\partial y}\varphi\bigg|}_{0} - \int\int dxdy2\pi\sum_i m_i\frac{\partial\varphi}{\partial x}\delta(y-y_i) - \underbrace{\frac{\partial\phi}{\partial x}\varphi\bigg|}_{0}\\
&+\int\int d^2x2\pi\sum_i m_i\frac{\partial\varphi}{\partial y}\delta(x-x_i)\Bigg]\\
&= 2\left[-\int dy2\pi\sum_i m_i\underbrace{\varphi|}_{0}\delta(y-y_i)+\int dx2\pi\sum_i m_i\underbrace{\varphi|}_{0}\delta(x-x_i)\right]\\
&= 0.
\end{aligned}
$$

because we are using periodic integrands, the linear terms are equal to zero.

Thus,

$$
\begin{aligned}
\beta H_{\mathrm{XY,low}} &= \frac{K}{2}\int d^2\vec{x}\left(\vec{\nabla}\theta\right)^2\\
&= \frac{K}{2}\int d^2x\left[(\vec{\nabla}\phi)^2+\left(\vec{\nabla}\times\hat{z}\varphi\right)^2\right]\\
&= \frac{K}{2}\int d^2x\left[(\vec{\nabla}\phi)^2+\left(\vec{\nabla}\varphi\right)^2\right].
\end{aligned}
$$

So

$$
\begin{aligned}
\beta H &= \frac{K}{2}\int d^2x\left(\vec{\nabla}\phi\right)^2+..\\
&= -\frac{K}{2}\int d^2x\phi\nabla^2\phi+..\\
&= -\frac{K}{2}\int d^2x\left(\sum_i m_i\log|\vec{x}-\vec{x}_i|\right)\left(2\pi\sum_j\delta(\vec{x}-\vec{x}_j)\right)+..\\
&= -2\pi^2\sum_{<i,j>}\frac{m_im_jK\log|\vec{x}_i-\vec{x}_j|}{2\pi}+..
\end{aligned}
$$

which is analogues to 2D coulomb gas

$$
-\pi^2\sum_{i<j}m_im_jK\frac{\log|\vec{x}-\vec{x}_j|}{2\pi}+\sum_{i=j}\beta\epsilon_o+..
$$

as the interaction coupling breaks down near the core; the self-interaction portion will just be labeled as core energy ϵ_o.
Say,

$$C\left(\vec{x}_i - \vec{x}_j\right) \equiv \frac{\log|\vec{x}_i - \vec{x}_j|}{2\pi},\ \text{Coulomb interaction}$$

$$\beta H = \frac{K}{2}\int d^2x\left(\vec{\nabla}\varphi\right)^2 + \sum_i \beta\epsilon^0_{m_i} - 4\pi^2\sum_{i<j} m_i m_j C\left(\vec{x}_i\vec{x}_j\right)K$$

So,

$$\mathcal{Z}_{XY} \approx \underbrace{\int D_\phi \exp\left\{-\frac{K}{2}\int d^2x(\nabla\varphi)^2\right\}}_{\mathcal{Z}\text{spin wave}}\underbrace{\sum_{m_i}\int\prod_i d^2x \exp\left\{-\sum_i \beta_i\epsilon^0_{m_i} + 4\pi^2 K\sum_{i<j} C\left(\vec{x}_i - \vec{x}_j\right)\right\}}_{\mathcal{Z}_Q,\ \text{grand canonical partition function of 2D gases with Coulomb interaction}}$$

$m_i = \pm 1$, $\sum_i m_i = 0$

We can also say $y \equiv e^{-\beta\epsilon^0_\pm}$

$$\mathcal{Z}_Q = \sum_N \frac{1}{\left(\frac{N}{2}\right)^2!}\int\prod_i d^2\vec{x}_i y^N \exp\left\{4\pi^2 K\sum_{i<j} m_i m_j C\left(\vec{x}_i - \vec{x}_j\right)\right\}$$

where $\left(\frac{N}{2}\right)!$ are the permutations of $+$ and $-$ charges. Now we have the coulomb gas paritition fuction abstracted from the XY model.

A.4.1 Maxwell Connections

To map the picture between connection Sine-gordon (broadly Maxwell's equation) and coloumb gas we need a voritce interaction term

$$\vec{\nabla}\theta = \vec{\nabla}\phi - \vec{\nabla}\times(\vec{z}\psi).$$

So

$$\begin{aligned}
\vec{\nabla}\times\vec{\nabla}\theta &= \underbrace{\vec{\nabla}\times\vec{\nabla}\phi}_{0} + \underbrace{\vec{\nabla}\times\vec{\nabla}\psi}_{0} - \vec{\nabla}(\vec{\nabla}\cdot\hat{z}\psi) - \nabla^2(\hat{z}\psi) \\
&= -\vec{\nabla}\underbrace{(\vec{\nabla}\cdot\hat{z}\psi)}_{\nabla_z\psi=0} - \nabla^2(\hat{z}\psi) \\
&= -\nabla^2(\hat{z}\psi),
\end{aligned}$$

for which $\vec{\nabla}\phi$ deals with spin waves while the latter deals with vortice interaction.
From a 2DXY-to-Sine-Gordon picture:

$$\begin{aligned}
\vec{E} = \vec{\nabla}\psi &\ \text{analogy between the electric field and the phase gradient;} \\
\vec{\nabla}\times\vec{E} = 0 &\ \text{since } \nabla^2\theta = 0 \text{ holds because the field is curl-free outside vortex cores;} \\
\vec{\nabla}\cdot E = \vec{\nabla}^2\psi = \rho_{\text{vortices}}, &\ \text{vortices as sources of the field is analogous to electric charges.}
\end{aligned}$$

A.5 A4. Sine-Gordon Connection

Sine-gordon with pertubation in g is

$$\beta H_{\rm sg} = \int d^2x \left[\frac{K'}{2}\left(\vec{\nabla}\varphi\right)^2 - g\cos\varphi\right]$$

and so using the discrete summation form of an exponent $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ and $g\cos\varphi = g\left[\frac{e^{i\varphi}+e^{-i\varphi}}{2}\right]$ if we say $\mathcal{Z}_g^0$ contains the hamiltonian where $g = 0$ (i.e., no perturbation) see that

$$\begin{aligned}
\mathcal{Z}_{\rm sg} = \int D_\varphi e^{-\beta H_{sg}} &= \int D_\varphi e^{-\tilde{S}_{\rm sg}} \sum_{N=0}^{\infty}\frac{\left(\int d^2x g\cos\varphi\right)^N}{N!} \\
&= \mathcal{Z}_{sg}^0\left(\sum_{N=0}^{\infty}\frac{(g/2)^N}{N!}\int\prod_{i=1}^{N}d^2x_i\sum_{\{m_i=\pm1\}}\frac{\exp\left\{-\int d^2x\frac{K^2}{2}\left(\vec{\nabla}\varphi\right)^2\right\}\exp\left\{i\sum_{i=1}^{N}m_i\varphi(\vec{x}_i)\right\}}{\mathcal{Z}_{sg}^0}\right) \\
&= \mathcal{Z}_{sg}^0\left(\sum_{N=0}^{\infty}\frac{(g/2)^N}{N!}\int\prod_{i=1}^{N}d^2x_i\sum_{\{m_i=\pm1\}}\left\langle\exp\left\{i\sum_{i=1}^{N}m_i\varphi(\vec{x}_i)\right\}\right\rangle\right).
\end{aligned}$$

See that

$$\begin{aligned}
&\left\langle\exp\left\{i\sum_{i=1}^{N}m_i\varphi(\vec{x}_i)\right\}\right\rangle \\
&= \exp\left\{-\frac{1}{2}\sum_{\alpha,\beta}m_\alpha m_\beta\left\langle\varphi(\vec{x}_\alpha)\varphi(x_\beta)\right\rangle\right\}.
\end{aligned}$$

Note for high T

$$\langle\varphi(\vec{x}_\alpha)\varphi(\vec{x}_\beta)\rangle = \frac{-1}{2\pi K}\log|\vec{x}_\alpha - \vec{x}_\beta| = \frac{-C(\vec{x}_\alpha-\vec{x}_\beta)}{K'}$$

$$\frac{\mathcal{Z}_{sg}}{\mathcal{Z}_{sg}^0} = \mathcal{Z}_Q.$$

So

$$\begin{aligned}
\mathcal{Z}_{Q,XY} &= \sum_N\frac{1}{(N/2)^2!}\int\prod_{i=1}^{N}d^2x_i\gamma^N\exp\left\{4\pi^2K\sum_{i,j}m_im_jC\left(\vec{x}_i-\vec{x}_j\right)\right\} \\
\mathcal{Z}_{Q,SG} &= \sum_N\frac{(1/2)^N}{N!}\int\prod_{i=1}^{N}d^2x_i\exp\left\{\sum_{\alpha,\beta}\frac{C\left(\vec{x}_\alpha-\vec{x}_\beta\right)}{2K'} - g\cos\varphi\right\}.
\end{aligned}$$

Clearly

$$K_{XY} = \frac{1}{8\pi^2 K_{SG}}.$$

Thus, the Sine-Gordon model partition function can also be expressed in terms of a Coulomb gas of charges $m_i = \pm 1$. Matching parameters shows the equivalence between the Coulomb gas from the XY vortices and that from the Sine-Gordon expansion.

A.6 A6. RG Flows via Integrating Out Short-Distance Pairs

To derive the RG flow, consider small vortex-antivortex pairs of size $r \approx a$. When integrating out these pairs (i.e., coarse-graining from rescaling), the fugacity y and coupling K change.

One finds for 2DXY RG flows:

$$\frac{dK^{-1}}{d\ell} = 4\pi^3 y^2, \quad \frac{dy}{d\ell} = (2 - \pi K)y.$$

These can be derived by carefully examining how adding a small vortex-antivortex pair affects correlation functions and by rescaling distances. The crucial steps involve:

1. Adding a pair of opposite charges at scale a
2. Computing their contribution to correlation functions
3. Rescaling coordinates
4. Matching coefficients to determine how K and y must change to keep the theory form-invariant.

The details of these steps are well-known but often summarized. Here I will provide a more explicit derivation below.

Detailed Derivation of RG Equations:

Now to derive one of the KG flows, see that

$$\left\langle e^{im\phi(\vec{x})} e^{-im\phi(\vec{y})} \right\rangle_0 = \exp\left\{ -\frac{m^2}{2} \left\langle (\phi(\vec{x}) - \phi(\vec{y}))^2 \right\rangle_0 \right\} = \exp\left\{ -\frac{m^2}{2} 2 * 2\pi K \log \frac{|\vec{x} - \vec{y}|}{a} \right\}$$
$$= \left(\frac{a}{|\vec{x} - \vec{y}|} \right)^{2\pi K m^2}.$$

We see there is a scaling dimension of $\pi K m^2$ for which is relevant under rescaling if $\pi K m^2 < d = 2$ and irrelevant if $\pi K m^2 > 2$ as for less than 2, perturbation becomes important and drives the system away from the original unperturbed point and vice-versa for bigger than 2.

Effective interaction between test charges

$$\left\langle e^{i\phi(\vec{x})}e^{-i\phi(\vec{x}')}\right\rangle$$

say is relabeled as $\langle V(\vec{x},\vec{x}')\rangle$ and perturbed with a normalization correction for which neutrality is conserved with even powers of integration (as odd ones vanish)

$$\begin{aligned}\langle V(\vec{x},\vec{x}')\rangle &= \frac{\left[\langle V(\vec{x},\vec{x}')\rangle_0 + y^2\int d^2yd^2y'\,\langle V(\vec{x},\vec{x}')V(\vec{y},\vec{y}')\rangle_0 + ...\right]}{1+y^2\int d^2yd^2y'\,\langle V(\vec{y},\vec{y}')\rangle_0 + ...}\\ &\approx \langle V(\vec{x},\vec{x}')\rangle_0\left[1+y^2\int d^2yd^2y'\exp\bigl\{-4\pi K^2C(\vec{y}-\vec{y}')\bigr\}(\exp\bigl(4\pi K^2D(\vec{x},\vec{x}',\vec{y},\vec{y}')\bigr)-1)+.\right].\end{aligned}$$

Note that

$$\begin{aligned}\langle V(\vec{x},\vec{x}')V(\vec{y},\vec{y}')\rangle_0 &= \left\langle e^{i\phi(\vec{x})}e^{-i\phi(\vec{x})}e^{i\phi(\vec{y})}e^{-i\phi(\vec{y}')}\right\rangle,\\ &= \exp\left\{-\sum_{\alpha<\beta}\langle\phi_\alpha\phi_\beta\rangle_0\, m_\alpha m_\beta\right\},\end{aligned}$$

where $\alpha = 1,2,3,4$ and $\langle\phi_\alpha\phi_\beta\rangle = 4\pi^2KC_{\alpha\beta}$ so we have six unique terms the correlation

$$\begin{gathered}C(y,y')\\ C(x,x')\\ C(y,x')\\ C(y',x)\\ C(x,y)\\ C(x',y')\end{gathered}$$

topological charges $(m_\alpha, m_\beta) = (+,-),(-,+),(-,-)$, or $(+,+)$ and we define the interaction between charges

$$\begin{aligned}D(x,x',y,y') &\equiv C(\vec{x}-\vec{y}) + C(\vec{x}'-\vec{y}') - C(\vec{y}'-\vec{x}) - C(\vec{y}-\vec{x}')\\ &= C(\vec{x}-\vec{R}-\frac{\vec{r}}{2}) + C(\vec{x}'-\vec{R}+\frac{\vec{r}}{2}) - C(\vec{x}'-\vec{R}-\frac{\vec{r}}{2}) - C(\vec{x}-\vec{R}+\frac{\vec{r}}{2})\\ &\approx C(\vec{x}-\vec{R}) - \frac{\vec{r}}{2}\cdot\nabla C(\vec{x}-\vec{R}) + C(\vec{x}-\vec{R}) + \frac{\vec{r}}{2}\cdot\nabla C(\vec{x}-\vec{R})\\ &- C(\vec{x}'-\vec{R}) + \frac{\vec{r}}{2}\cdot\nabla C(\vec{x}'-\vec{R}) - C(\vec{x}'-\vec{R}) - \frac{\vec{r}}{2}\cdot\nabla C(\vec{x}'-\vec{R})\\ &= -\vec{r}\cdot\vec{\nabla}C(\vec{x}-\vec{R}) + \vec{r}\cdot\vec{\nabla}C(\vec{x}'-\vec{R}) + \mathcal{O}(r^3).\end{aligned}$$

Also see that

$$\exp\bigl\{4\pi^2KD(x,x',y,y')\bigr\}-1 \approx -4\pi^2K\vec{r}\cdot\vec{\nabla}\left(C(\vec{x}-\vec{R})-C(\vec{x}'-\vec{R}\right)+8\pi^4K^2\left(\vec{r}\cdot\vec{\nabla}(...)\right)^2+\mathcal{O}(r^3).$$

Note r is an odd function so under periodic integration

$$\int d^2rr\cdot\nabla f(P) = 0$$

so,

$$\langle V(\vec{x},\vec{x}')\rangle = \exp\{-4\pi^2 K C(\vec{x}-\vec{x}')\}\left[1+2\pi y^2\int drr\exp\{-4\pi^2 KC(\vec{r})\}8\pi^4K^2r^2\frac{1}{2}F(\vec{x},\vec{x}')+...\right],$$

where

$$\begin{aligned}
F(\vec{x},\vec{x}') &= \int d^2R\left(\vec{\nabla}\left(C(\vec{x}-\vec{R})-C(\vec{x}'-\vec{R})\right)\right)^2 \\
&= \int d^2R\left((\vec{\nabla}C(\vec{x}-\vec{R}))^2 - 2\vec{\nabla}C(\vec{x}-\vec{R})\cdot\vec{\nabla}C(\vec{x}'-\vec{R}) + (\vec{\nabla}C(\vec{x}'-\vec{R}))^2\right) \\
&= \int d^2R\left(C(\vec{x}-\vec{R})\underbrace{\nabla^2C(\vec{x}-\vec{R})}_{\delta(\vec{x}'-R)} - 2C(\vec{x}-\vec{R})\underbrace{\nabla^2C(\vec{x}'-\vec{R})}_{\delta(\vec{x}'-\vec{R}')} + C(\vec{x}'-\vec{R})\underbrace{\nabla^2C(\vec{x}'-\vec{R})}_{\delta(\vec{x}'-\vec{R})}\right) \\
&= 2C(\vec{x}-\vec{x}')-2C(0).
\end{aligned}$$

Thus

$$\begin{aligned}
\langle V(\vec{x},\vec{x}')\rangle &= \exp\{-4\pi^2KC(\vec{x}-\vec{x}')\}\exp\left\{16\pi^5K^2y^2C(\vec{x}-\vec{x}')\int_a^\infty drr^3\left(\frac{r}{a}\right)^{-2\pi K}\right\}+\mathcal{O}(y^3) \\
&\equiv \exp\{-4\pi^2K_{eff}C(\vec{x}-\vec{x}')\},
\end{aligned}$$

where effective stiffness K_{eff} is a renormalized measure of a system's resistance to phase fluctuations or deformations, incorporating the effects of thermal fluctuations and topological defects such as vortices. In systems like the 2D XY model or Sine-Gordon theory, the bare stiffness K determines the energy cost of phase gradients, but thermal excitations and defect unbinding reduce this stiffness at larger scales. Physically, K_{eff} describes how the system's rigidity changes when observed on different length scales, reflecting the interplay between order and disorder. Near the KT tranistion, K_{eff} exhibits critical behavior: it remains finite below the transition temperature, where vortex-antivortex pairs are bound, but vanishes above it as free vortices destroy long-range phase coherence. Thus, K_{eff} serves as a key parameter to understand the system's macroscopic behavior in the presence of perturbations or thermal effects. As will be seen later.

Now see that

$$
\begin{aligned}
K_{eff} &= K - 4\pi^3 K^2 y^2 a^{2\pi k} \int_a^\infty dr r^{3-2\pi K} + \mathcal{O}(y^4) \\
&= K\left(1 - 4\pi^3 K y^2 a^{2\pi K}\left[\frac{r^{4-2\pi K}}{4-2\pi K}\right]\Bigg|_a^{la}\right) - 4\pi^3 K^2 y^2 a^{2\pi K}\int_{la}^\infty dr r^{3-2\pi K} + \mathcal{O}(y^4) \\
&= K\left(1 - 4\pi^3 K y^2 a^{2\pi K}\left[\frac{(la)^{4-2\pi K} - a^{4-2\pi K}}{4-2\pi K}\right]\right) - 4\pi^3 K^2 y^2 a^{2\pi K}\int_{la}^\infty dr r^{3-2\pi K} + \mathcal{O}(y^4) \\
&\approx K\left(1 - 4\pi^3 K y^2 a^{2\pi K}\left[\frac{(1+(4-2\pi K)\delta\rho)a^{4-2\pi K} - a^{4-2\pi K}}{4-2\pi K}\right]\right) \\
&- 4\pi^3 K^2 y^2 a^{2\pi K}\int_{la}^\infty dr r^{3-2\pi K} \\
&= K\left(1 - 4\pi^3 K y^2 a^4 \delta\rho\right) - 4\pi^3 K^2 y^2 a^{2\pi K} l^{4-2\pi K}\int_a^\infty \underbrace{dr' r'^{3-2\pi K}}_{r\to r'l} \\
&= K\underbrace{\left(1 - 4\pi^3 K y^2 a^4 \delta\rho\right)}_{K'} - 4\pi^3 K^2 a^{2\pi K}\underbrace{y^2 l^{4-2\pi K}}_{y'^2}\int_a^\infty dr' r'^{3-2\pi K} \\
&\approx K\underbrace{\left(1 - 4\pi^3 K y^2 a^4 \delta\rho\right)}_{K'} - 4\pi^3 K^2 a^{2\pi K}\underbrace{y^2(1+(4-2\pi K)\delta\rho)}_{y'^2}\int_a^\infty dr' r'^{3-2\pi K}
\end{aligned}
$$

For which we're saying $\frac{r^{4-2\pi K}}{4-2\pi K}$ arises naturally when integrating terms like $r^{3-2\pi K}$ over the radial length r as part of coarse-graining (renormalization) $\int_a^\infty = \int_a^{la} + \int_{la}^\infty$ and a rescaling of $r \to rl$ for infinitesimally small logarithmic change $\delta\rho$ so $l = e^{\delta\rho} \approx (1+\delta\rho)$.

See that for a=1,

$$
\begin{aligned}
y' &= y\left(1 + (2-\pi K)\,\delta\rho\right), \\
K' &= K\left(1 - 4\pi^3 K y^2 \delta\rho\right).
\end{aligned}
$$

Using $\dfrac{dK^{-1}}{d\rho} = -\dfrac{1}{K^2}\dfrac{dK}{d\rho}$ we see that after integrating out a small fluctuation, the vortex frugacity y' and stiffness K' change with scaling as follows

$$
\begin{aligned}
\frac{dK'^{-1}}{d\rho} &= 4\pi^3 y^2, \\
\frac{dy'}{d\rho} &= (2-\pi K)\,y.
\end{aligned}
$$

A.7 A7. Invariance and Flow Lines

Let's define $x = K^{-1} - \frac{\pi}{2}$, where near the critical point $x \approx K^{-1}$, so that

$$
\begin{aligned}
K &= \frac{1}{K^{-1}} \\
&= \frac{1}{\frac{\pi}{2} + x} \\
&= \frac{1}{\frac{\pi}{2}\left(1 + \frac{2x}{\pi}\right)} \\
&= \frac{2}{\pi\left(1 + \frac{2x}{\pi}\right)} \\
&\approx \frac{2}{\pi}\left(1 - \frac{2x}{\pi}\right).
\end{aligned}
$$

So,

$$
\begin{aligned}
\frac{dy}{d\rho} &\approx \left(2 - \pi\left(\frac{2}{\pi}\left(1 - \frac{2x}{\pi}\right)\right)\right) y \\
&= \left(2 - \left(2\left(1 - \frac{2x}{\pi}\right)\right)\right) y \\
&= \frac{4xy}{\pi}.
\end{aligned}
$$

Now see that

$$
\begin{aligned}
\frac{d}{d\rho}\left(x^2 - \pi^4 y^2\right) &= 2x\frac{dx}{d\rho} - 2\pi^4 y \frac{dy}{d\rho} \\
&= 2x\frac{dK^{-1}}{d\rho} - 2\pi^4 y\left[\frac{4}{\pi}xy\right] \\
&= 2x4\pi^3 y^2 - 2\pi^4 y\left[\frac{4}{\pi}K^{-1}y\right] \\
&= 8K^{-1}\pi^3 y^2 - 8\pi^3 y^2 K^{-1} \\
&= 0.
\end{aligned}
$$

So

$$
\left(x^2 - \pi^4 y^2\right) = C,
$$

where C is a constant.

Thus, the RG flows follow hyperbolas in the (x, y) plane as seen in Figure 3. One fixed line is $y = 0$, $K > 2/\pi$. At $K = 2/\pi$, y is marginal. For $K < 2/\pi$, y flows to larger values, indicating proliferation of vortices.

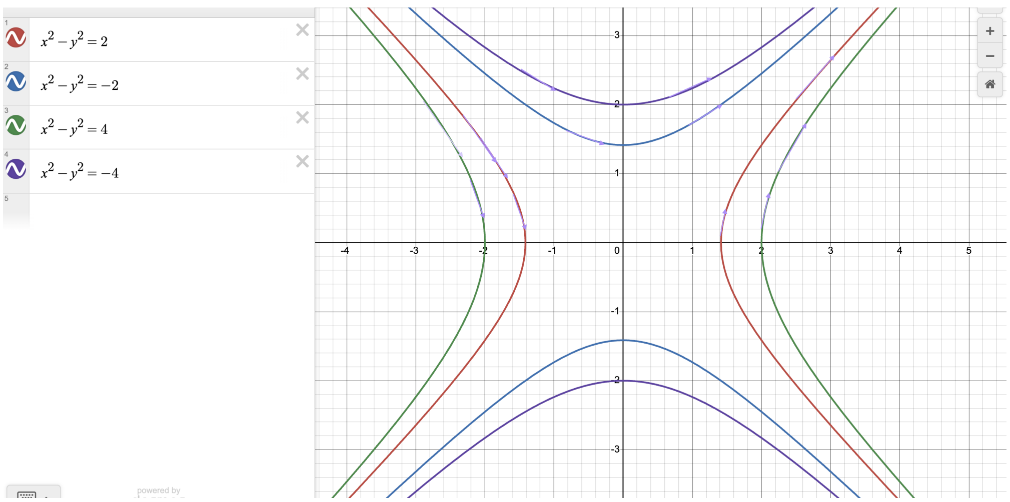

Figure 3: Simplified KT RG flow for C>0, C<0